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Exponential Hermite–Euler Splines

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Recently I. J. Schoenberg studied the cardinal splines that interpolate the function λ^x at the integers, where λ is a complex number. This paper deals with cardinal splines which together with their successive derivatives interpolate λ^x and its successive derivatives at the integers.

INTRODUCTION

Let n, r be positive integers such that $n \geqslant 2r - 1$. The class $\mathscr{S}_{n,r}$ of cardinal splines of degree n with integer knots of multiplicity r consists of the functions $S(x)$ such that $S(x)$ is a polynomial of degree n in each of the intervals $[\nu, \nu + 1]$ $(\nu = 0, \pm 1, \pm 2, \ldots)$ and $S(x) \in C^{n-r}(-\infty, \infty)$.

In an interesting paper [4] Schoenberg studied the cardinal splines $S_n(x; \lambda)$, called the exponential Euler splines, that interpolate the function λ^x at the integers, where λ is a complex number (see also [7]). These exponential Euler splines $S_n(x; \lambda)$ are extremely useful (see [5, 7]). It turns out that $S_n(x; \lambda)$ are "periodic extensions" of the exponential Euler polynomials $A_n(x; \lambda)$ introduced by Euler [1]. These polynomials are generated by the relation

$$\frac{\lambda - 1}{\lambda - e^z} e^{xz} = \sum_{n=0}^{\infty} \frac{A_n(x; \lambda)}{n!} z^n. \tag{1}$$

The essential properties of $A_n(x; \lambda)$ are given in [4].

This paper deals with cardinal splines $S_{n,r}(x; \lambda) \in \mathscr{S}_{n,r}$ which together with their successive derivatives interpolate the function λ^x and its successive derivatives at the integers, i.e.,

$$S_{n,r}^{(\rho)}(\nu) = \lambda^\nu (\log \lambda)^\rho \qquad (\rho = 0, 1, \ldots, r - 1)\ \forall \text{ integers}. \tag{2}$$

In Section 1 we introduce the polynomials $A_{n,r,s}(x; \lambda)$ from which the splines $S_{n,r,s}(x; \lambda)$ are constructed in Section 2. The representations of $S_{n,r,s}(x; \lambda)$ in terms of B-splines are given in Section 3. In Section 4 we study the behavior of $S_{n,r}(x; \lambda)$ as n tends to infinity, and in the last section we give a complete proof of the convergence theorem for the case $r = 2$.

1. The Polynomial $A_{n,r,s}(x;\lambda)$

Let $s = 0, 1,\ldots, r-1$ be a fixed integer and set

$$A_{n,r,s}(x;\lambda) = \begin{vmatrix} \frac{A_n(0;\lambda)}{n!} & \frac{A_{n-1}(0;\lambda)}{(n-1)!} & \cdots & \frac{A_{n-s+1}(0;\lambda)}{(n-s+1)!} & \frac{A_n(x;\lambda)}{n!} & \frac{A_{n-s-1}(0;\lambda)}{(n-s-1)!} & \cdots & \frac{A_{n-r+1}(0;\lambda)}{(n-r+1)!} \\ \frac{A_{n-1}(0;\lambda)}{(n-1)!} & \frac{A_{n-2}(0;\lambda)}{(n-2)!} & \cdots & \frac{A_{n-s}(0;\lambda)}{(n-s)!} & \frac{A_{n-1}(x;\lambda)}{(n-1)!} & \frac{A_{n-s-2}(0;\lambda)}{(n-s-2)!} & \cdots & \frac{A_{n-r}(0;\lambda)}{(n-r)!} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{A_{n-r+1}(0;\lambda)}{(n-r+1)!} & \frac{A_{n-r}(0;\lambda)}{(n-r)!} & \cdots & \frac{A_{n-r-s+2}(0;\lambda)}{(n-r-s+2)!} & \frac{A_{n-r+1}(x;\lambda)}{(n-r+1)!} & \frac{A_{n-r-s}(0;\lambda)}{(n-r-s)!} & \cdots & \frac{A_{n-2r+2}(0;\lambda)}{(n-2r+2)!} \end{vmatrix} \tag{1.1}$$

where $A_n(x;\lambda)$ are the exponential Euler polynomials. From the relation $A_n'(x;\lambda)/n! = A_{n-1}(x;\lambda)/(n-1)!$ it is easy to see that

$$A_{n,r,s}^{(s)}(0;\lambda) = H_r(A_n(0;\lambda)/n!), \tag{1.2}$$

where $H_r(a_n)$ denotes the Hankel determinant of order r given by

$$H_r(a_n) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_{n-r+1} \\ a_{n-1} & a_{n-2} & \cdots & a_{n-r} \\ \vdots & \vdots & & \vdots \\ a_{n-r+1} & a_{n-r} & \cdots & a_{n-2r+2} \end{vmatrix}.$$

Using the relation

$$H_r(\Pi_n(\lambda)/n!) = (-1)^{[r/2]}\, C(n,r)\, \Pi_{n,r}(\lambda), \tag{1.3}$$

where

$$C(n,r) = \frac{1!\,2!\cdots(r-1)!}{n!(n-1)!\cdots(n-r+1)!}$$

(see [3]), it follows that

$$A_{n,r,s}^{(s)}(0;\lambda) = (-1)^{[r/2]+(r-1)(n-r+1)}\,\frac{C(n,r)\,\Pi_{n,r}(\lambda)}{(\lambda-1)^{n-r+1}}, \tag{1.4}$$

where $\Pi_n(\lambda) = (\lambda - 1)^n A_n(0; \lambda)$. Further, from the properties $A_n^{(\rho)}(1; \lambda) = \lambda A_n^{(\rho)}(0; \lambda)$ $(\rho = 0, 1,..., n-1)$, it is easy to check that $A_{n,r,s}(x; \lambda)$ satisfy the relations

$$A_{n,r,s}^{(\rho)}(1; \lambda) = \lambda A_{n,r,s}^{(\rho)}(0; \lambda) \qquad (\rho = 0, 1,..., n-r), \tag{1.5}$$

$$\left.\begin{aligned} &A_{n,r,s}^{(\rho)}(1; \lambda) = A_{n,r,s}^{(\rho)}(0; \lambda) = 0 \qquad (\rho = 0, 1,..., r-1, \rho \neq s), \\ &A_{n,r,s}^{(s)}(0; \lambda)/H_r(A_n(0; \lambda)/n!) = 1, \end{aligned}\right\} \tag{1.6}$$

provided $\lambda \neq 1$ and λ is not a zero of $\Pi_{n,r}(\lambda)$, an assumption which we shall impose throughout this paper.

2. The Exponential Hermite–Euler Splines $S_{n,r}(x; \lambda)$

Let us define a function $S_{n,r,s}(x; \lambda)$ $(s = 0, 1,..., r-1)$ such that

$$\left.\begin{aligned} S_{n,r,s}(x; \lambda) &= A_{n,r,s}(x; \lambda)/H_r(A_n(0; \lambda)/n!) \qquad (0 \leqslant x \leqslant 1) \\ S_{n,r,s}(x+1; \lambda) &= \lambda S_{n,r,s}(x; \lambda) \qquad \forall \text{ real } x. \end{aligned}\right\} \tag{2.1}$$

It follows from (1.5) and (1.6) that $S_{n,r,s}(x) \in C^{n-r}(-\infty, \infty)$ and

$$\left.\begin{aligned} &S_{n,r,s}^{(\rho)}(\nu, \lambda) = 0 \qquad (\rho = 0, 1,..., r-1, \rho \neq s), \\ &S_{n,r,s}^{(s)}(\nu, \lambda) = \lambda^\nu \qquad (\nu = 0, \pm 1, \pm 2,...), \end{aligned}\right\} \tag{2.2}$$

so that it is cardinal spline belonging to the class

$$\mathscr{S}_{n,r}^{(s)} = \{S(x) \in \mathscr{S}_{n,r} : S^{(\rho)}(\nu) = 0 \ \forall \text{ integers}, \rho = 0, 1,..., r-1, \rho \neq s\}.$$

When $r = 1$ (in which case $s = 0$), $S_{n,1,0}(x; \lambda) = S_n(x; \lambda)$ are the exponential Euler splines considered by Schoenberg [4].

Now, set

$$S_{n,r}(x; \lambda) = \sum_{s=0}^{r-1} (\log \lambda)^s \, S_{n,r,s}(x; \lambda) \qquad (x \in R). \tag{2.3}$$

The following theorem is an easy consequence of (2.2).

Theorem 2.1. *The spline functions $S_{n,r}(x; \lambda)$ belong to $\mathscr{S}_{n,r}$ and satisfy the interpolatory conditions*

$$S_{n,r}^{(\rho)}(\nu; \lambda) = (\log \lambda)^\rho \, \lambda^\nu \qquad (\rho = 0, 1,..., r-1) \qquad \textit{for all } \nu = 0, \pm 1, \pm 2,..., \tag{2.4}$$

3. Representation of $S_{2m-1,r,s}(x;\lambda)$ in Terms of B-Splines

The B-splines for cardinal Hermite interpolation, denoted by $N_s(x)$ ($s = 0, 1,..., r-1$), were introduced by Schoenberg and Sharma [6]. These B-splines belong to the spaces $\mathscr{S}^{(s)}_{2m-1,r}$, have support in $(-(m-r+1), (m-r+1))$, and satisfy the interpolatory properties

$$\begin{aligned} N_s^{(s)}(\nu) &= C_\nu \qquad (\nu = -(m-r),..., (m-r)), \\ &= 0 \qquad \text{otherwise,} \end{aligned} \tag{3.1}$$

where C_ν are the coefficients of the monic reciprocal polynomial $\Pi_{2m-1,r}(\lambda) = \sum_{\nu=0}^{2m-2r} C_{\nu-(m-r)}\lambda^\nu$.

It was shown in [2] that the 'polynomial component of the spline $s!\,\lambda^{(m-r)}\sum_{-\infty}^{\infty}\lambda^\nu N_s(x-\nu)$ in $[0, 1]$ is given explicitly by the determinant

$$\begin{vmatrix}
x^s & 1 & \binom{s}{1} & \cdots (1-\lambda) & 0 \cdots 0 & 0 & \cdots & 0 \\
x^r & 1 & \binom{r}{1} & \cdots & \binom{r}{r-1}(1-\lambda) & 0 & \cdots & 0 \\
x^{r+1} & 1 & \binom{r+1}{1} & & & (1-\lambda)\, 0 \cdots & & 0 \\
\vdots & \vdots & \vdots & & & & \ddots & \vdots \\
 & & & & & & & 0 \\
x^{2m-r-1} & 1 & \binom{2m-r-1}{1} & \cdots & \cdots & \cdots & \cdots & (1-\lambda) \\
\vdots & \vdots & \vdots & & & & & \vdots \\
x^{2m-2} & 1 & \binom{2m-2}{1} & \cdots & \cdots & \cdots & \cdots & \binom{2m-2}{2m-r-1} \\
x^{2m-1} & 1 & \binom{2m-1}{1} & \cdots & \cdots & \cdots & \cdots & \binom{2m-1}{2m-r-1}
\end{vmatrix}. \tag{3.2}$$

By an argument similar to that in [2], using the properties (1.5) and (1.6), it can be shown that the polynomial

$$\frac{s!\,\Pi_{2m-1,r}(\lambda)\, A_{2m-1,r,s}(x;\lambda)}{H_r(A_{2m-1}(0;\lambda)/(2m-1!))} \qquad (x \in [0, 1])$$

is also given by (3.2), provided λ is not a zero of $\Pi_{2m-1,r}(\lambda)$. Hence

$$\frac{A_{2m-1,r,s}(x;\lambda)}{H_r(A_{2m-1}(0;\lambda)/(2m-1)!)} = \frac{1}{\Pi_{2m-1,r}(\lambda)}\sum_{-\infty}^{\infty}\lambda^{(m-r)+\nu}N_s(x-\nu) \quad (x \in [0, 1]). \tag{3.3}$$

From (2.1) and (3.3) we easily deduce the following

THEOREM 3.1. *The exponential Hermite–Euler spline* $S_{2m-1,r,s}(x;\lambda)$ *is expressible in terms of the B-spline* $N_s(x)$ *by*

$$S_{2m-1,r,s}(x;\lambda) = \frac{1}{\Pi_{2m-1,r}(\lambda)} \sum_{\infty}^{\infty} \lambda^{\nu} N_s(x + (m-r) - \nu). \tag{3.4}$$

4. CONVERGENCE OF EXPONENTIAL HERMITE–EULER SPLINES

When $r = 1$, Schoenberg [4] proved that $\lim_{n\to\infty} S_n(x;\lambda) \to \lambda^x$ uniformly for x belonging to a finite interval, if λ is a nonnegative complex number. In general we have the following result.

THEOREM 4.1. *If* λ *is a complex number which is not of sign* $(-1)^r$, *then*

$$\lim_{n\to\infty} S_{n,r}^{(\rho)}(x;\lambda) = (\log\lambda)^{\rho}\,\lambda^x \qquad (\rho = 0, 1, 2,..., r-1). \tag{4.1}$$

uniformly for x *belonging to a finite interval.*

The results of the above theorem follow from the corresponding results for the functions $S_{n,r,s}(x;\lambda)$. In order to state the latter results we write $\lambda = |\lambda|\, e^{i\alpha}$ and $\lambda_k = \log|\lambda| + i(\alpha + 2\pi k)$ $(k = 0, \pm 1, \pm 2,...)$. In [4] it was shown that the exponential Euler polynomial $A_n(x;\lambda)$ has the following expansion.

$$A_n(x;\lambda)/n! = (\lambda - 1)\,\lambda^{-1}\,\lambda^{x'} \sum_{-\infty}^{\infty} e^{2\pi i k x}/\lambda_k^{n+1}. \tag{4.2}$$

If we define a numerical sequence $\{\mu_k\}$ $(k = 0, 1, 2,...)$ by

$$\mu_0 = \lambda_0\,,\quad \mu_1 = \lambda_{-1}\,,\quad \mu_2 = \lambda_1\,,\quad \mu_3 = \lambda_{-2}\,,\quad \mu_4 = \lambda_2\,,...,\tag{4.3}$$

and the corresponding sequence of functions $\{u_k(x)\}$ $(k = 0, 1, 2,...)$ by

$$u_0(x) = 1,\quad u_1(x) = e^{-2\pi i x},\quad u_2(x) = e^{2\pi i x},\quad u_3(x) = e^{-2\pi 2 i x},...,\tag{4.4}$$

then (4.2) can be written as

$$A_n(x;\lambda)/n! = (\lambda - 1)\,\lambda^{-1}\lambda^{x} \sum_{0}^{\infty} u_k(x)/\mu_k^{n+1}. \tag{4.5}$$

Next, we introduce the notation $V(a_0, a_1,..., a_{r-1})$ to stand for the Vandermonte determinant

$$V(a_0, a_1,..., a_{r-1}) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{r-1} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{r-1} \\ \vdots & \vdots & & & \vdots \\ 1 & a_{r-1} & a_{r-1}^2 & \cdots & a_{r-1}^{r-1} \end{vmatrix} \tag{4.6}$$

and let $V_s(a_0, a_1, ..., a_{r-1}; u(x))$ $(s = 0, 1, ..., r - 1)$ be the determinants obtained from (4.6) by replacing the sth column by the column vector $(u_0(x), u_1(x), u_2(x), ..., u_{r-1}(x))^T$. For each $s = 0, 1, ..., r - 1$, define

$$\phi_s(x; \lambda) = \lambda^x \frac{V_s(\mu_0, \mu_1, ..., \mu_{r-1}; u(x))}{V(\mu_0, \mu_1, ..., \mu_{r-1})}. \tag{4.7}$$

The behavior of the exponential Hermite–Euler splines $S_{n,r,s}(x; \lambda)$ as $n \to \infty$ is described by the following

THEOREM 4.2. *Let $\lambda = |\lambda| e^{i\alpha}$. The following relation holds uniformly for x belonging to a finite interval:*

$$\lim_{n\to\infty} S_{n,r,s}^{(\rho)}(x; \lambda) = \phi_s^{(\rho)}(x; \lambda) \cdot \quad (\rho = 0, 1, ..., r - 1) \tag{4.8}$$

for $-\pi < \alpha < \pi$ when r is odd, and for $0 < \alpha < 2\pi$ when r is even.

The proofs of Theorem 4.1 and 4.2 involve tedious determinantal manipulations. We shall give a complete proof only for the case $r = 2$.

5. CONVERGENCE FOR THE CASE $r = 2$

When $r = 2$, the results of Theorem 4.2 can be expressed in a simple form in terms of the functions

$$\alpha(x) = \lambda^x e^{-\pi i x}(\sin \pi x)/\pi, \tag{5.1}$$

$$\beta(x) = \lambda^x - (\log \lambda)\, \alpha(x). \tag{5.2}$$

More precisely we have

THEOREM 5.1. *Let $\lambda = |\lambda| e^{i\alpha}$. If $0 < \alpha < 2\pi$, the following relations hold uniformly for all x belonging to finite interval:*

$$\lim_{n\to\infty} S_{n,2,0}^{(\rho)}(x; \lambda) = \beta^{(\rho)}(x) \qquad (\rho = 0, 1), \tag{5.3}$$

and

$$\lim_{n\to\infty} S_{n,2,1}^{(\rho)}(x; \lambda) = \alpha^{(\rho)}(x) \qquad (\rho = 0, 1). \tag{5.4}$$

Clearly, the results of Theorem 4.1 for the case $r = 2$ follow from (5.3) and (5.4). A proof of Theorem 5.1 depends on the following lemma.

LEMMA 5.2. *Let* $\lambda = |\lambda| e^{i\alpha}$ *and* $\lambda_k = \log|\lambda| + i(\alpha + 2\pi k)$. *The following relations hold uniformly for all x in* [0, 1]:

$$\lim_{n\to\infty} \lambda_0^{n+1} A_n(x; \lambda)/n! = (\lambda - 1)\,\lambda^{-1}\lambda^x \qquad (-\pi < \alpha \leqslant \pi), \tag{5.5}$$

$$\begin{aligned}&\lim_{n\to\infty} \lambda_1^{n+1}\{A_{n-1}(x; \lambda)/n - 1)! - \lambda_0 A_n(x; \lambda)/n!\} \\ &\quad = (\lambda + 1)\,\lambda^{-1}\,\lambda^x e^{2\pi i x}(2\pi i) \qquad (-\pi < \alpha < 0),\end{aligned} \tag{5.6}$$

$$\begin{aligned}&\lim_{n\to\infty} \lambda_{-1}^{n+1}\{A_{n-1}(x; \lambda)/(n - 1)! - \lambda_0 A_n(x; \lambda)/n!\} \\ &\quad = (\lambda - 1)\,\lambda^{-1}\lambda^x e^{-2\pi i x}(-2\pi i) \qquad (0 < \alpha \leqslant \pi).\end{aligned} \tag{5.7}$$

Proof. Using the expansion (4.2) we have

$$\lambda_0^{n+1} A_n(x; \lambda)/n! = (\lambda - 1)\,\lambda^{-1}\lambda^x \sum_{-\infty}^{\infty} e^{2\pi i k x}(\lambda_0/\lambda_k)^{n+1}. \tag{5.8}$$

Since $|\lambda_0| < |\lambda_k|\ \forall k \neq 0$, (5.5) follows from (5.8). Also from (4.2) we have

$$\begin{aligned}&\lambda_1^{n+1}\{A_{n-1}(x; \lambda)/(n - 1)! - \lambda_0 A_n(x; \lambda)/n!\} \\ &\quad = (\lambda - 1)\,\lambda^{-1}\lambda^x \sum_{k\neq 0} (\lambda_k - \lambda_0)(\lambda_1/\lambda_k)^{n+1}\, e^{2\pi i k x}.\end{aligned} \tag{5.9}$$

If $-\pi < \alpha < 0$, $|\lambda_1| < |\lambda_k|\ \forall k \neq 0, 1$, and (5.6) follows from (5.9). The limit (5.7) is proved in the same way. ∎

Proof of Theorem 5.1. We shall prove only the relation

$$\lim_{n\to\infty} S_{n,2,0}(x; \lambda) = \beta(x). \tag{5.10}$$

The rest are proved in the same way.

We can write

$$\lambda_0^{n+1}\lambda_{-1}^{n+1} A_{n,2,0}(x; \lambda) = \begin{vmatrix} \lambda_0^{n+1} A_n(x; \lambda)/n! & \lambda_0^{n+1} A_{n-1}(0; \lambda)/(n-1)! \\ \lambda_{-1}^{n+1}\{A_{n-1}(x; \lambda)/(n - 1)! - \lambda_0 A_n(x; \lambda)/n!\} & \lambda_{-1}^{n+1}\{A_{n-2}(0; \lambda)/(n - 2)! - \lambda_0 A_{n-1}(0; \lambda)/(n - 1)!\} \end{vmatrix}.$$

If $0 < \alpha \leqslant \pi$, it follows from (5.5) and (5.7) that

$$\lambda_0^{n+1}\lambda_{-1}^{n+1} A_{n,2,0}(x; \lambda) \to (\lambda - 1)^2\,\lambda^{-2}\lambda^x(\lambda_{-1} - \lambda_0)(\lambda_{-1} - \lambda_0 e^{-2\pi i x}). \tag{5.11}$$

Hence from (2.1) and (5.11) we have

$$\lim_{n\to\infty} S_{n,2,0}(x;\lambda) = \lambda^x(\lambda_{-1} - \lambda_0 e^{-2\pi i x})/(\lambda_{-1} - \lambda_0)$$
$$= \lambda^x\{1 - (\log\lambda)(1 - e^{-2\pi i x})/2\pi i\} \qquad (0 < \alpha \leqslant \pi). \quad (5.12)$$

Similarly,

$$\lim_{n\to\infty} S_{n,2,0}(x;\lambda) = \lambda^x\{1 + (\log\lambda)(1 - e^{2\pi i x})/2\pi i\}$$
$$= \lambda^x e^{2\pi i x}\{1 - (\log\lambda + 2\pi i)(1 - e^{-2\pi i x})/2\pi i\} \quad (5.13)$$
$$(-\pi < \alpha < 0).$$

Combining (5.12) and (5.13) we obtain

$$\lim_{n\to\infty} S_{n,2,0}(x;\lambda) = \lambda^x\{1 - (\log\lambda)(1 - e^{-2\pi i x})/2\pi i\} \quad (5.14)$$

when $\lambda = |\lambda| e^{i\alpha}$ for $0 < \alpha < 2\pi$, from which (5.10) follows. ∎

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References

1. L. Euler, "Institutionis calculi differentialis," Vol. II, Petersburg, 1755.
2. S. L. Lee, *B*-splines for cardinal Hermite interpolation, *J. Linear Algebra Appl*, to appear.
3. S. L. Lee and A. Sharma, Cardinal Lacunary interpolation by *g*-splines, I, The characteristic polynomials, *J. Approximation Theory*, to appear.
4. I. J. Schoenberg, Cardinal interpolation and spline functions, IV. The exponential Euler splines, *in* "Proceedings of the Oberwolfach Conference" (August 1971); *ISNM* **201** (1972), 382–404.
5. I. J. Schoenberg, Notes on spline functions, III. On the convergence of interpolating cardinal splines as their degree tends to infinity, *J. Analyse Math.* **16** (1973), 87–93.
6. I. J. Schoenberg and A. Sharma, Cardinal interpolation and spline functions, V, The *B*-splines for cardinal Harmite interpolation, *J. Linear Algebra Appl.* **7** (1973), 1–42.
7. I. J. Schoenberg, "Cardinal Spline Interpolation," SIAM, Philadelphia, 1973.